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The Weak Sequential Continuity of the Metric Projection in L_p Spaces Over Separable Nonatomic Measure Spaces

JOSEPH M. LAMBERT

The Pennsylvania State University, York Campus, York, Pennsylvania 17403

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1. INTRODUCTION

Let X be a reflexive Banach space and M a Chebyshev subspace of X. The metric projection (nearest point map) supported by M is denoted by $P(M)$. A long standing problem in approximation theory is to determine necessary and sufficient conditions that $P(M)$ be norm continuous for all Chebyshev subspaces M contained in X. Whenever M is a finite dimensional Chebyshev subspace, a compactness argument shows that $P(M)$ is norm continuous. Necessary and sufficient conditions that $P(M)$ be norm continuous whenever M has finite codimension have been given in [4, Theorem 14] and [1, Theorem 8]. In the case of nonreflexive Banach spaces, examples have been given in [7, p. 87], [5, Example 4], [1, Theorem 36] and [9, Theorem 4] of finite codimensional subspaces which do not have norm continuous metric projections.

In [4] a study of the weak sequential continuity of the metric projection was initiated in the hopes of possibly finding a parallel attack to the problems concerning the norm continuity of the metric projection. The following necessary and sufficient condition for the weak sequential continuity of the metric projection was found [4, Theorem 11].

THEOREM (Holmes). *Let X be a reflexive Banach space, M a Chebyshev subspace. Then the following are equivalent*:

(a) *$P(M)$ is weakly sequentially continuous.*
(b) *$\{x \in X \mid P(M)(x) = 0\}$ is weakly sequentially closed.*

Weak sequential continuity was studied, since in the proof of the above theorem, an application of the uniform boundedness theorem was used, thus insisting that sequences be used and not nets. Some questions concerning weak continuity of the metric projection have been studied in [10].

A simple corollary to the above theorem shows that whenever X is a reflexive Banach space and M is a Chebyshev subspace of finite codimension, then $P(M)$ is weakly sequentially continuous, whenever $P(M)$ is norm continuous. In particular, this occurs in the L_p spaces, $1 < p < \infty$. When the dimension of M is finite, the question of the weak sequential continuity of the metric projection is left open.

This paper is devoted to answering the question concerning the weak sequential continuity of the metric projection onto a subspace of finite dimension in an L_p space. The work exhibits the weak sequential closure of the kernel of the metric projection, in order to apply the criterion of Holmes. It is found that for any finite dimensional subspace of an L_p space over a separable nonatomic measure space, the weak sequential closure of the metric projection is the entire L_p space. Hence the metric projection is not weakly sequentially continuous.

In another paper, we will discuss the results pertaining to L_p spaces over measure spaces which contain atoms. Such spaces will admit weakly sequentially continuous metric projections.

In this work, if we denote a Banach space as X, X^* denotes the space of continuous linear functionals on X. The symbol $(.,.)$ is used to denote the duality relationship of X with X^*. It will sometimes be used to denote an ordered pair in a product space. Its use will be clear from the context. $U(X)$ will be the closed unit ball of X, that is, $U(X) = \{x \text{ in } X \mid \| x \| \leqslant 1\}$. $S(X)$ will denote the closed unit sphere: $S(X) = \{x \text{ in } X \mid \| x \| = 1\}$. If M is a subspace of X, $S(X) \cap M$ will be denoted by $S(M)$. M^0 will designate the set of linear functionals in X^* which are identically zero on M. The convergence of x_n to x in the weak topology will be noted by $x_n \rightharpoonup x$, and the norm convergence of x_n to x will be denoted $x_n \to x$.

$\mathbf{R}$ will signify the real number field. $\mathbf{R}^+$ will denote the set of positive real numbers. $A \triangle B$ is the set $(A \backslash B) \cup (B \backslash A)$. χ_A denotes the characteristic function of A. The real valued function $\operatorname{sgn}(\cdot)$ is defined via $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(x) = x/|x|$, for $x \neq 0$. The set valued function $\operatorname{supp}(\cdot)$ is defined on a function space $\mathbf{R}^X$ as follows: if f is in $\mathbf{R}^X$, then $\operatorname{supp}(f) = \{x \text{ in } X \mid f(x) \neq 0\}$. The abbreviation clspan denotes closed span. All other notation and terminology follows [2].

DEFINITION 1.1. Let M be a subspace of a Banach space X. M is a *Chebyshev* subspace if x in X implies $\inf\{\| x - m \| \mid m \text{ in } M\}$ is attained at a unique element of M. Let M be a Chebyshev subspace of a Banach space X. Then $P(M): X \to M$ is the *metric projection* of X on M defined via $\inf\{\| x - m \| \mid m \text{ in } M\} = \| x - P(M)(x)\|$. Let $k(M) = \{x \text{ in } X \mid P(M)(x) = 0\}$ Let $K(M)$ be the weak sequential closure of $k(M)$. We recall that if X is a Banach space, M a Chebyshev subspace, then

$$k(M) = \bigcup_{L \in S(M^0)} \{x \text{ in } X \mid L(x) = \| x \|\}.$$

DEFINITION 1.2. Let X be a Banach space. Let $f: X \to X$. Then f is *weakly sequentially continuous* if and only if $x_n \rightharpoonup x$ implies $f(x_n) \rightharpoonup f(x)$.

2. THE L_p CASE

This section contains the major result of this paper. First, we formulate the results for the measure space of the real number field, Borel subsets, and Lebesgue measure: $(\mathbf{R}, B, m)$. We are interested in finding the finite dimensional subspaces M in $L_p(\mathbf{R}, B, m)$ such that $P(M)$ is weakly sequentially continuous. Theorem 2.1 shows that no such M exists in $L_p(\mathbf{R}, B, m)$ for $p \neq 2$. This section concludes with a sketch extending the results from $(\mathbf{R}, B, m)$ to a separable nonatomic measure space.

DEFINITION 2.1. Let (X, W, μ) be a measure space with a positive real valued measure. $L_p(X, W, \mu)$ is the space of real valued μ-measurable functions satisfying, f is in $L_p(X, W, \mu)$ if and only if $(\int_X |f|^p \, d\mu)^{1/p} < \infty$, where $1 \leqslant p < \infty$.

It is well known that whenever $1 < p < \infty$, then $L_p(X, W, \mu)$ is a reflexive, rotund, smooth Banach space with $(L_p)^* = L_q$ where $1/p + 1/q = 1$ [6; p. 360], [2; p. 89].

For the remainder of this paper let T denote the duality map from $L_p(X, W, \mu)$ to $L_q(X, W, \mu)$ given by $Tf(x) = |f(x)|^{p-1} \operatorname{sgn}(f(x))_i$ The following two lemmas are elementary properties of this map T.

LEMMA 2.1. *Let a_i be real numbers. Let g_i be in $S(L_p(X, W, \mu))$ with* supp g_i *being pairwise disjoint, $i = 1,..., k$. If $r = \sum_{i=1}^k a_i Tg_i$, then $\| r \|_q = [\sum_{i=1}^k | a_i |^q]^{1/q}$.*

Proof. $\| r \|_q = [\int | \sum a_i Tg_i |^q]^{1/q} = [\sum | a_i |^q \int | Tg_i |^q]^{1/q} = [\sum | a_i |^q]^{1/q}$ Q.E.D.

Note that in the above proof we have used the fact that Tg_i is in $S(L_q(X, W, \mu))$ and that supp Tg_i are the pairwise disjoint, $i = 1,..., k$.

LEMMA 2.2. *Let r be defined as above. Then r attains its norm on* $S(\operatorname{span}(g_i \mid i = 1,..., k))$.

Proof. Let $s = \sum_{i=1}^k b_i g_i$. Then

$$(s, r) = \Big(\sum_i b_i g_i, \sum_j a_j Tg_j\Big) = \sum_i a_i b_i (g_i, Tg_i) = \sum_i a_i b_i .$$

To satisfy the proposition we must find b_i, $i = 1,..., k$ such that

$$\sum_i a_i b_i = \left(\sum_i |a_i|^q\right)^{1/q} \quad \text{and} \quad \left(\sum_i |b_i|^p\right)^{1/p} = 1.$$

But since $\{a_i \mid i = 1,..., k\}$ is in $l_q(k)$ there exists a sequence $\{b_i \mid i = 1,..., k\}$ in $l_p(k)$ satisfying the desired conditions. Q.E.D.

The following definitions, although formidable, help simplify statements of propositions and their proofs.

DEFINITION 2.2. *A in W is p-discontinuous* if there exists $\{g_n\}$ in $L_p(X, W, \mu)$ such that

(1) $\|g_n\| = 1$, $\mu(A \triangle \operatorname{supp} g_n) = 0$;

(2) $\int_A g_n \, d\mu = 0$; there is $\epsilon > 0$, $C < \infty$ such that $\epsilon \leqslant |g_n(x)| < C$ for almost all x in A;

(3) $g_n \rightharpoonup 0$, and

(4) $Tg_n \rightharpoonup C_A \chi_A$ with $C_A \neq 0$.

W is p-discontinuous if A is p-discontinuous for all A in W.

DEFINITION 2.3. Let M be a subspace of $L_q(X, W, \mu)$, $1 < q < \infty$, *A in W is (q, M)-discontinuous* if it satisfies (a) or (b):

(a) χ_A is in M^0.

(b) there exist $\{g_n\}$ in $L_p(X, W, \mu)$, $1 < p < \infty$, $1/p + 1/q = 1$, such that

(1) $\|g_n\| = 1$, $\mu(A \triangle \operatorname{supp} g_n) = 0$;

(2) g_n is in M^0 for all n;

(3) $g_n \rightharpoonup 0$; and

(4) $Tg_n \rightharpoonup C_A \chi_A$, $C_A \neq 0$.

Let M be a subspace of $L_q(X, W, \mu)$, $1 < q < \infty$. Then *W is (q, M)-discontinuous* if A is (q, M)-discontinuous for all A in W.

PROPOSITION 2.1. *Let M be a closed subspace of $L_q(X, W, \mu)$. If W is (q, M)-discontinuous, then every simple function is an element of $K(M)$.*

Proof. Let $r = \sum_{j=1}^{s} a_j \chi_{A_j}$ be an arbitrary simple function. Since W is (q, M)-discontinuous there is for every j a sequence $\{g_n(A_j)\}$ in M^0 with $Tg_n(A_j) \rightharpoonup C_{A_j} \chi_{A_j}$. Choose b_j such that $C_{A_j} b_j = a_j$. Let $r_n = \sum_{j=1}^{s} b_j Tg_n(A_j)$.

By Lemma 2.2, there is a linear functional $v_n = \sum_{j=1}^{s} d_j g_n(A_j)$ such that $(v_n, r_n) = \| r_n \|$ and $\| v_n \| = 1$. Hence r_n is in $k(M)$. But

$$r_n = \sum_{j=1}^{s} b_j T g_n(A_j) \rightharpoonup \sum_{j=1}^{s} a_j \chi_{A_j} = r.$$

Therefore r is in $K(M)$. Q.E.D.

The following lemma is highly technical, but is needed to obtain the major results of this section.

LEMMA 2.3. *Let* $\{g_n\}$, $\{f_n\}$ *be elements of* $L_p(X, W, \mu)$ *and* $A \in W$ *be such that*

(1) $\mu(A \triangle \operatorname{supp} f_n) = 0$ *for all* n;

(2) *there exists* $0 < C < \infty$, $\epsilon > 0$ *such that* $\epsilon \leqslant | g_n(x)| < C$ *for almost all* x *in* A, *for all* n;

(3) $| g_n |^{p-1} \operatorname{sgn}(g_n) \rightharpoonup g$; *and*

(4) $f_n \to 0$ *uniformly except on a set of zero measure.*

Then $| g_n - f_n |^{p-1} \operatorname{sgn}(g_n - f_n) \rightharpoonup g$.

Proof. Recall that if f_n, f are elements of $L_q(X, W, \mu)$, $1 < q < \infty$, then $f_n \rightharpoonup f$ if and only if $\{f_n\}$ is bounded and $\int_A f_n \, d\mu \to \int_A f \, d\mu$ for all A in W such that $\mu(A) < \infty$. Since $f_n \to 0$ uniformly except on a set of zero measure and $| g_n(x)| < \epsilon$ for all n and for almost all x in A, there exists $N(\epsilon)$ such that for all $n > N(\epsilon)$, $\operatorname{sgn}(g_n - f_n) = \operatorname{sgn}(g_n)$ except on a set of zero measure. Thus for all $n > N(\epsilon)$,

$$\begin{aligned} &| \, | g_n - f_n |^{p-1} \operatorname{sgn}(g_n - f_n) - | g_n |^{p-1} \operatorname{sgn}(g_n)| \\ &\quad = | \, | g_n - f_n |^{p-1} - | g_n |^{p-1} \, | \end{aligned}$$

except on a set of zero measure. Let B be an arbitrary set of finite measure in W. Then

$$\begin{aligned} &\left| \int_B (| g_n - f_n |^{p-1} \operatorname{sgn}(g_n - f_n) - g) \, d\mu \right| \\ &\quad \leqslant \left| \int_B (| g_n - f_n |^{p-1} \operatorname{sgn}(g_n - f_n) - | g_n |^{p-1} \operatorname{sgn}(g_n)) \, d\mu \right| \\ &\quad\quad + \left| \int_B (| g_n |^{p-1} \operatorname{sgn}(g_n) - g) \, d\mu \right|. \end{aligned}$$

The second integral can be made arbitrarily small since $|g_n|^{p-1}\operatorname{sgn}(g_n) \rightharpoonup g$. We now consider the first integral. By our initial remarks we have

$$\left| \int_B (|g_n - f_n|^{p-1} \operatorname{sgn}(g_n - f_n) - |g_n|^{p-1} \operatorname{sgn}(g_n))\, d\mu \right|$$
$$\leqslant \int_B \left| |g_n - f_n|^{p-1} - |g_n|^{p-1} \right| d\mu \qquad \text{for all} \quad n > N(\epsilon).$$

Case 1: $1 < p < 2$. By using the inequality of [8, p. 155] we have $\int_B \left| |g_n - f_n|^{p-1} - |g_n|^{p-1} \right| d\mu \leqslant \int_B |f_n|^{p-1}\, d\mu$. But $f_n \to 0$ uniformly except of a set of zero measure. Hence the first integral can be made arbitrarily small.

Case 2: $2 \leqslant p \leqslant \infty$. Since for all $n > N(\epsilon)$, $|f_n(x)| < \epsilon$ a.e. and $g_n(x)| < C$ for almost all x in A, we have $|g_n(x)|$ and $|g_n(x) - f_n(x)|$ are elements of the finite interval $[0, C + \epsilon]$ for almost all x in A. By using the fact that $|\cdot|^{p-1}$ is a convex function for $p \geqslant 2$, we have

$$\left| |g_n(x) - f_n(x)|^{p-1} - |g_n(x)|^{p-1} \right| \leqslant C_1 |f_n(x)|$$

for almost all x in A, where C_1 depends on ϵ, p, and C. Since $\mu(A \triangle \operatorname{supp} f_n) = 0$ for all n, we apply this inequality to the first integral obtaining $\int_B \left| |g_n - f_n|^{p-1} - |g_n|^{p-1} \right| d\mu \leqslant C_1 \int_B |f_n|\, d\mu$ where C_1 depends on ϵ, p, and C. But again $f_n \to 0$ uniformly except on a set of zero measure. Hence the first integral can be made arbitrarily small.

It remains to show that $|g_n - f_n|^{p-1}$ is L_q bounded. But since $\mu(A \triangle \operatorname{supp} f_n) = 0$ for all n and $|g_n|^{p-1} \operatorname{sgn} g_n \rightharpoonup g$, we have $|g_n - f_n|^{p-1}$ is L_q bounded on $X \backslash A$ and by previous remarks $|g_n(x) - f_n(x)|$ is bounded for almost all x in A and hence $|g_n(x) - f_n(x)|^{p-1}$ is L_q bounded on A.

Hence $|g_n - f_n|^{p-1} \operatorname{sgn}(g_n - f_n) \rightharpoonup g$. Q.E.D.

PROPOSITION 2.2. *Let* $M = \operatorname{span}(z_i \mid i = 1,..., m)$ *be an m-dimensional subspace of* $L_q(X, W, \mu)$, $1 < q < \infty$. *Let A be an element of W such that* $\mu(A) < \infty$. *If A is q-discontinuous then A is* (q, M)*-discontinuous.*

Proof. Let $M_A = \operatorname{span}(z_i \chi_A \mid i = 1,..., m)$.

Case 1: $\dim M_A = v$ *where* $1 < v \leqslant m$. Since $\dim M_A = v$, there exist t_i, $i = 1,..., v$ such that $\operatorname{supp} t_i$ is in A and $M_A = \operatorname{span}(t_i \chi_A \mid i = 1,..., v)$. Let $\{g_n\}$ be the sequence satisfying the conditions of Definition 2.2. This exists since A is q-discontinuous. Let $c(n, t_i) = \int_A g_n t_i\, d\mu$. Note that $c(n, t_i)$ is well defined since g_n is bounded almost everywhere on A and t_i is in L_q. Let $M_j = \operatorname{span}(t_i \chi_A \mid i \neq j, i = 1,..., v)$. M_j is contained in $L_q(A, \Sigma(A), \mu)$, a finite measure space, where $\Sigma(A)$ is the σ-field induced by the set A. $M_j \subset L_1(A, \Sigma(A), \mu)$. $M_j \neq \{0\}$ by hypothesis of Case 1. Since M_j is a closed subspace of $L_1(A, \Sigma(A), \mu)$ and $t_j\chi_A$ is not an element of M_j, there exists an h_j in $L_\infty(A, \Sigma(A), \mu)$ such that $(t, h_j) = 0$ for all t in M_j and $(t_j\chi_A, h_j) = 1$ [2; p. 64]. Hence h_j is in $L_p(A, \Sigma(A), \mu)$. Thus h_j is in $L_\infty(A, \Sigma(A), \mu) \cap L_p(A, \Sigma(A), \mu)$. Set $\hat{g}_n = g_n - \sum_{j=1}^{v} c(n, t_j)\, h_j\chi_A$. $\{\hat{g}_n\}$ is in $L_p(X, W, \mu)$. Also $\int_A \hat{g}_n t_i\, d\mu = \int_A g_n t_i\, d\mu - \sum_{j=1}^{v} c(n, t_j) \int_A t_i h_j\, d\mu = c(n, t_i) - c(n, t_i) = 0$. Thus $\hat{g}_n$ is in $M_A{}^0$ and hence in M^0, since $\mu(A \triangle \operatorname{supp} g_n) = 0$. We remark that $c(n, t_i) \to 0$, since $g_n \rightharpoonup 0$. Hence $\sum_{j=1}^{v} | c(n, t_j)| \to 0$ as $n \to \infty$. Let $f_n = \sum_{j=1}^{v} c(n, t_j)\, h_j\chi_A$. Since h_j is in $L_\infty(A, \Sigma(A), \mu)$, we have $f_n \to 0$ uniformly except on a set of zero measure. Hence $\hat{g}_n$ can be written as $\hat{g}_n = g_n - f_n$. Clearly $\| \hat{g}_n \|_p \leqslant \| g_n \|_p + \| f_n \|_p$. Since $f_n \to 0$ uniformly except on a set of zero measure and $\| g_n \| = 1$ for all n, $\| \hat{g}_n \| \to 1$. Thus there exists N such that for all $n > N(1/2)$, $\| \hat{g}_n \|_p > 1/2$. For $n > N$, define $v_n = \hat{g}_n / \| \hat{g}_n \|$. Without loss of generality we assume $N = 0$. To show $v_n \rightharpoonup 0$, it suffices to show $\hat{g}_n \rightharpoonup 0$. But $\hat{g}_n = g_n - f_n$, where $g_n \rightharpoonup 0$ and $f_n \rightharpoonup 0$. Hence $\hat{g}_n \rightharpoonup 0$. To show $Tv_n \rightharpoonup C_A\chi_A$; $C_A \neq 0$, it suffices to show $| \hat{g}_n |^{p-1} \operatorname{sgn}(\hat{g}_n) \rightharpoonup C_g\chi_A$, $C_g \neq 0$. But $\hat{g}_n = g_n - f_n$ satisfies the conditions of Lemma 2.3. Hence $Tv_n \rightharpoonup C_A\chi_A$, $C_A \neq 0$. Thus $\{v_n\}$ satisfies condition (b) of Definition 2.3 of A being (q, M)-discontinuous.

Case 2: $\dim(\operatorname{span}(t_i\chi_A \mid i = 1,..., m)) = 1$. Then $\operatorname{span}(t_i\chi_A \mid i = 1,..., m) = \operatorname{span}(t\chi_A)$ for some t in $L_q(X, W, \mu)$. If $\int_A t\, d\mu = 0$, χ_A is in M^0 and A satisfies condition (a) of Definition 2.3 and hence is (q, M)-discontinuous. If $\int_A t\, d\mu \neq 0$, set $\hat{g}_n = (1/\mu(A))\, \chi_A - (\int_A g_n t\, d\mu / \int_A t\, d\mu)\, \chi_A$. But $\hat{g}_n$ has the same properties of the $\hat{g}_n$ discussed in Case 1 and hence we have the result for Case 2. Q.E.D.

PROPOSITION 2.3. *Let B be a ring. Let $(X, \Sigma(B), \mu)$ be a σ-finite measure space. Let M be a closed subspace of $L_q(X, \Sigma(B), \mu)$. If for all A in B, A is (q, M)-discontinuous, then every simple function is an element of $K(M)$.*

Proof. Let $r = \sum_{n=1}^{m} a_n\chi_{A_n}$ be a simple function with A_n in B. By the proof of Proposition 2.1, r is in $K(M)$. Let $s = \sum_{n=1}^{m} a_n\chi_{B_n}$ where B_n is in $\Sigma(B)$. Let $\epsilon > 0$ be given. Then by [3, p. 56] there exists A_n in B such that $\mu(A_n \triangle B_n) \leqslant \epsilon$. Hence

$$\| r - s \|_q^q = \int_X \left| \sum_{n=1}^{m} a_n\chi_{B_n} - \sum_{n=1}^{m} a_n\chi_{A_n} \right|^q d\mu$$

$$\leqslant \int_X \sum_{n=1}^{m} | a_n |^q \, | \chi_{B_n} - \chi_{A_n} |^q \, d\mu \leqslant \sum_{n=1}^{m} | a_n |^q \, \epsilon^q.$$

Thus $\| r - s \|_q$ can be made arbitrarily small. Since $K(M)$ is norm closed and simple functions of the form of r are in $K(M)$, we have s is in $K(M)$. Q.E.D.

The following measure theoretic concepts and facts can be found in [3, Sec. 41].

DEFINITION 2.4. Let (X, W, μ) be a finite positive measure space. A *partition* of an element E in W is a finite set P of disjoint elements of W whose union is E. Let $P = (E_1, ..., E_n)$ be a partition of E in W. Define the *norm of* P to be $\| P \| = \max(\mu(E_i) \mid i = 1, ..., n)$. A sequence of partitions $\{P_n\}$ is called *dense* if to every element E of W and to every $\epsilon > 0$, there exists an n and E_0 in W such that $E_0 = \bigcup_{i \in F} E_i$, and $\mu(E \triangle E_0) < \epsilon$, where the E_i are in the partition P_n and F is a finite index set.

Fact 2.1. If Y is the unit interval in $\mathbf{R}$, B the class of Borel subsets of Y, m Lebesgue measure on B and if $\{Q_n\}$ is a sequence of partitions of Y into intervals such that $\lim_{n\to\infty} \| Q_n \| = 0$, then $\{Q_n\}$ is dense.

The next two results are concerned with the particular measure space $(\mathbf{R}, B, m)$ where B is the class of all Borel sets of $\mathbf{R}$ and m is Lebesgue measure.

PROPOSITION 2.4. *Let I be a finite interval in R in the measure space* $(\mathbf{R}, B, m)$. *Then I is p-discontinuous,* $1 < p \neq 2 < \infty$.

Proof. Let P_n be the canonical partition of I into 2^n distinct disjoint intervals $I(j, n)$ where $j = 1, ..., 2^n$, with $m(I(j, n)) = 1/2^n m(I)$. We note that if $I(j, n+1)$ is in P_{n+1} then $I(j, n+1) \subset I(i, n)$ for some index i. Also the sequence $\{P_n\}$ in dense by Fact 2.1. For each n and for each $I(j, n)$, we form the two disjoint intervals $I(j, n, 1)$ and $I(j, n, 2)$ contained in $I(j, n)$ such that $m(I(j, n, 1)) = 3m(I(j, n, 2))$. Define the functions f_n on I as follows for each n: $f_n(x) = \sum_{i=1}^{2^n} \chi_{I(i,n,1)}^{(x)} - 3\chi_{I(i,n,2)}^{(x)}$. Note that $\{f_n\}$ is in $L_p(\mathbf{R}, B, n)$ since

$$\|f_n\|_p^p = \sum_{i=1}^{2^n} (m(I(i, n, 1)) + 3^p m(I(i, n, 2))$$

$$= \sum_{i=1}^{2^n} (3 + 3^p)\, m(I(i, n, 2)) = ((3 + 3^p)1/4)\, m(I).$$

We normalize f_n, setting $g_n = f_n/\|f_n\|$. Clearly $\| g_n \| = 1$ and g_n satisfies condition (1) of Definition 2.2. Since supp $g_n = I$, we have

$$\int_R g_n\, dm = \int_I g_n\, dm = (1/\|f_n\|) \int_I f_n\, dm$$

$$= (1/\|f_n\|) \sum_{i=1}^{2^n} (m(I(i, n, 1)) - 3m(I(i, n, 2))) = 0.$$

It is clear that $4/((3+3^p)\,m(I)) \leqslant |g_n(x)| \leqslant (4\cdot 3)/((3+3^p)\,m(I))$ for all x in I. Hence $\{g_n\}$ satisfies condition (2) of Definition 2.2.

We note that $\{f_n\}$ are uniformly bounded on I. Denote this bound by M_g. To show $g_n \rightharpoonup 0$, it suffices to show $f_n \rightharpoonup 0$. Hence it is sufficient to show that given any Borel set D in B with $m(D) < \infty$ and given $\epsilon > 0$, there exists an $N(\epsilon, D)$ such that $|\int_D f_n\,dm| < \epsilon$ for all $n > N(\epsilon, D)$.

Case 1: $D \cap I = \varnothing$ *is trivial.*

Case 2: $E = D \cap I \neq \varnothing$. Since $E \subset I$, we have by the implication of Fact 2.1 that there exists an $N(\epsilon, E)$ such that there exists an $E_0 = \bigcup_{j\in F} I(j, N)$ with $m(E \triangle E_0) < \epsilon$, where F is a finite index set. Thus $|\int_E f_n\,dm| \leqslant |\int_E f_n\,dm - \int_{E_0} f_n\,dm| + |\int_{E_0} f_n\,dm|$. By construction $\int_{E_0} f_n\,dm = 0$ for all $n > N(\epsilon, E)$. But

$$\left|\int_E f_n\,dm - \int_{E_0} f_n\,dm\right| \leqslant \left|\int_{E\mathrm{W}E_0} |f_n|\,dm\right| \leqslant M_g m(E \triangle E_0) < M_g\epsilon.$$

Setting $\epsilon' = M_g \cdot \epsilon$ we have $|\int_D f_n\,dm| < \epsilon'$ for every $n > N(\epsilon', D) = N(\epsilon', E)$. Hence $\{g_n\}$ satisfies condition (3) of Definition 2.2.

It remains to show that $Tg_n \rightharpoonup C_I\chi_I$, $C_I \neq 0$, in order to show $\{g_n\}$ satisfies condition (4) of Definition 2.2.

$$\begin{aligned} Tg_n &= T(f_n/\|f_n\|) = (1/\|f_n\|^{p-1})\,|f_n|^{p-1}\operatorname{sgn}(f_n) \\ &= (1/\|f_n\|^{p-1}) \left|\sum_{i=1}^{2^n} \chi_{I(i,n,1)} - 3\chi_{I(i,n,2)}\right|^{p-1} \operatorname{sgn}\left(\sum_{i=1}^{2^n} \chi_{I(i,n,1)} - 3\chi_{I(i,n,2)}\right) \\ &= (1/\|f_n\|^{p-1}) \sum_{i=1}^{2^n} \chi_{I(i,n,1)} - 3^{p-1}\chi_{I(i,n,2)} \end{aligned}$$

because of the disjoint supports of the $\chi_{I(i,n,j)}$. Hence

$$\begin{aligned} \int_R Tg_n\,dm = \int_I Tg_n\,dm &= (1/\|f_n\|^{p-1}) \int_I \left(\sum_{i=1}^{2^n} \chi_{I(i,n,1)} - 3^{p-1}\chi_{I(i,n,2)}\right) dm \\ &= (1/\|f_n\|^{p-1}) \sum_{i=1}^{2^n} (m(I(i,n,1)) - 3^{p-1} m(I(i,n,2))) \\ &= (1/\|f_n\|^{p-1})(3(1 - 3^{p-2})\,m(I))/4. \end{aligned}$$

Since $p \neq 2$, $\int_I Tg_n\,dm = C_I m(I)$; $C_I \neq 0$.

We remark that since $\|f_n\|^{p-1}$ is a constant for all n, to show $Tg_n \rightharpoonup C_I\chi_I$, it suffices to show $Tf_n \rightharpoonup K_I\chi_I$ where K_I is the nonzero constant such that $\int_I Tf_n\,dm = K_I m(I)$. Also we note that $\{Tf_n\}$ are uniformly bounded on I. Denote this bound by M_{Tg}.

To show $Tf_n \rightharpoonup K_I\chi_I$, it suffices to show that for any Borel set D of finite measure and given $\epsilon > 0$, there exists an $N(\epsilon, D)$ such that $|\int_D Tf_n\,dm - K_I \int_D \chi_I\,dm| < \epsilon$, for all $n > N(\epsilon, D)$.

Case 1: $D \cap I = \varnothing$ *is trivial.*

Case 2: $D \cap I = E \neq \varnothing$. For the same reasons as before we obtain an $N(\epsilon, E)$ and E_0 such that $m(E \triangle E_0) < \epsilon$. Thus

$$\begin{aligned}\left| \int_E Tf_n\,dm - \int_E K_I\chi_I\,dm \right| &\leqslant \left| \int_E Tf_n\,dm - \int_{E_0} Tf_n\,dm \right| + \left| \int_{E_0} Tf_n\,dm - K_I \int_E \chi_I\,dm \right| \\ &\leqslant M_{Tg} m(E \triangle E_0) + |K_I m(E_0) - K_I m(E)|,\end{aligned}$$

since $\int_{E_0} Tf_n\,dm = K_I m(E_0)$ by construction. Therefore

$$\left| \int_E Tf_n\,dm - \int_E K_I\chi_I\,dm \right| \leqslant (M_{Tg} + |K_I|)\, m(E \triangle E_0) \leqslant (M_{Tg} + |K_I|) \cdot \epsilon.$$

Thus $Tf_n \rightharpoonup K_I\chi_I$. Hence $\{g_n\}$ satisfies the conditions of Definition 2.2 and I is p-discontinuous. Q.E.D.

The following result characterizes $K(M)$ for any finite dimensional subspace M of $L_q(\mathbf{R}, B, m)$.

THEOREM 2.1. *Let M be a finite dimensional subspace of $L_q(\mathbf{R}, B, m)$, $1 < q \neq 2 < \infty$. Then $K(M) = L_q(\mathbf{R}, B, m)$.*

Proof. Let I be any interval in B. Proposition 2.4, I is q-discontinuous. By Proposition 2.2, I is (q, M)-discontinuous. Since B is the σ-field generated by finite unions of disjoint intervals, we have every simple function is in $K(M)$ by Proposition 2.3. But $K(M)$ is norm closed and the simple functions are dense in $L_q(\mathbf{R}, B, m)$. Q.E.D.

To conclude this section it should be noted that the results can be extended from $L_q(\mathbf{R}, B, m)$ to $L_q(X, W, \mu)$ where (X, W, μ) is a separable nonatomic measure space. A reference for the measure theoretic concepts is [3, Ch. VIII]. The technique is to note that there exists an isomorphism between every separable nonatomic normalized measure algebra (W, μ) and the measure algebra of the unit interval [3, Section 41, Theorem C]. Since the isomorphism is a measure preserving transformation we can relate the integrals over the measure spaces by [3, Section 39, Theorem C]. It can then be shown that X is q-discontinuous by taking the sequence $\{f_n\}$ which satisfies Definition 2.2 for the unit interval and checking that the sequence $\{f_n \cdot L\}$ satisfies Definition 2.2

for the set X, where $L: (X, W, \mu) \to (I, B, m)$ is the isomorphism of the measure algebras. This process is reasonably straightforward and yields no new insight. So in particular, we have that given an arbitrary set A of finite measure in W then A is q-discontinuous. But by Proposition 2.2, A is (q, M)-discontinuous and by Proposition 2.3 every simple function is in $K(M)$ and such functions are dense in $L_q(X, W, \mu)$. Hence we obtain

THEOREM 2.2. *Let (X, W, μ) be a separable nonatomic measure space. Let M be a finite dimensional subspace of $L_q(X, W, \mu)$, $1 < q \neq 2 < \infty$, then $K(M) = L_q(X, W, \mu)$. Hence the metric projection onto M is not weakly sequentially continuous.*

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REFERENCES

1. E. CHENEY AND D. WULBERT, The existence and unicity of best approximations, *Math. Scan.* **24** (1969), 113–140.
2. N. DUNFORD AND J. SCHWARTZ, "Linear Operators I," Interscience, New York, 1958.
3. P. HALMOS, "Measure Theory," Van Nostrand, New York, 1950.
4. R. B. HOLMES, On the continuity of best approximation operators, contributed to the Symposium on Infinite Dimensional Topology, Baton Rouge, La., 1967.
5. R. B. HOLMES AND B. KRIPKE, Smoothness of approximation, *Mich. Math. J.* **15** (1968), 225–248.
6. G. KOTHE, "Topological Vector Spaces I," Springer Verlag, New York, 1969.
7. J. LINDENSTRAUS, Extension of compact operators, *Mem. Amer. Math. Soc.* **48** (1964).
8. M. LOEVE, "Probability Theory," Van Nostrand, New York, 1963.
9. P. MORRIS, Metric projections onto subspaces of finite codimension, *Duke Math. J.* **35** (1968), 799–808.
10. C. KOTTMAN AND B. L. LIN, The weak continuity of metric projections, *Mich. Math. J.* **17** (1970), 401–404.